Homework #8

CEE 225: Dynamics of Structures

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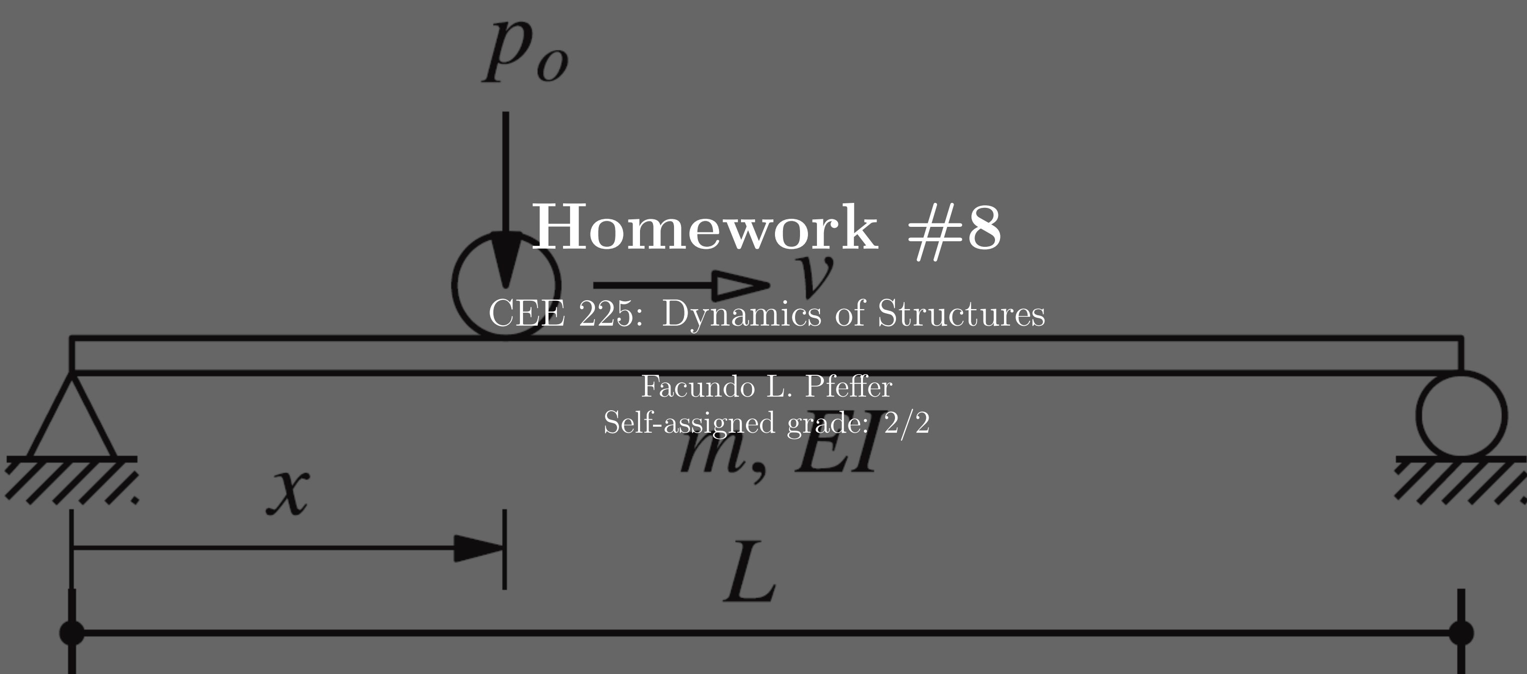
Self-assigned grade: 2/2



Self-Grading Declaration

Based on the guidelines provided, **I self-assign a grade of 2/2 for this homework**. I have made a legitimate effort to complete 100% of the problems.

Declaration of Computational and Artificial Intelligence Assistance

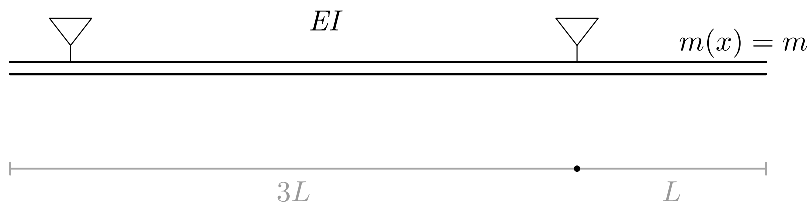
To streamline algebraic manipulations and avoid error-prone, cumbersome calculations, I used Wolfram Mathematica (Wolfram Language) to evaluate definite integrals, perform symbolic differentiation, and simplify expressions appearing in this work.

Explanatory text in the appendices was drafted with assistance from an artificial intelligence (AI) system (ChatGPT, OpenAI) to clarify intermediate steps (splitting, expanding, differentiating, squaring, integrating, and combining) and to improve LaTeX formatting. The mathematical results themselves (numerical values, closed-form expressions, and conclusions) were verified by me through independent checks, including continuity of the piecewise mode shape at $X = 3L$, dimensional consistency, and comparison against the Wolfram Mathematica outputs.

1 Problem 1

Assignment 1

By Rayleigh's method determine the natural vibration frequency of the uniform beam shown in Fig. 1. Assume that the shape function is given by the deflections due to a vertical force applied at the free end.



By problem requirement, the mode shape is obtained from the deflected shape due to a load applied at the free end, which can be obtained by solving the ODE with a linear elastic beam model:

$$y'' = \frac{M}{EI}$$

The function of the mode shape (derived in Section .0) is:

$$\Psi(X) = \begin{cases} \dfrac{P}{18EI} X\left(9L^2 - X^2\right) \\ \\ \dfrac{P}{6EI}\left(3L - X\right)\left(X^2 - 9LX + 12L^2\right), \quad 3L \leq X \leq 4L. \end{cases}$$

The mass and stiffness functions $m(X)$ and $EI(x)$ are constant across the beam. The generalized mass $\tilde{m}$ and stiffness $\tilde{k}$ are[1]:

$$\tilde{m} = m \int_0^{4L} \Psi^2(X) dX$$
$$\tilde{k} = EI \int_0^{4L} [\Psi''(X)]^2 dX$$

Solving the integrations (see **??** for details):

$$\tilde{m} = \frac{37}{35}\frac{m\,P^2\,L^7}{EI^2}, \qquad \tilde{k} = \frac{4}{3}\frac{P^2\,L^3}{EI}.$$

Finally, the natural frequency of the system is:

$$\omega_n^2 = \frac{\tilde{k}}{\tilde{m}} = \frac{\frac{4}{3}\dfrac{P^2L^3}{EI}}{\frac{37}{35}\dfrac{mP^2L^7}{EI^2}} = \frac{140}{111}\frac{EI}{m\,L^4}. \to \omega_n \approx 1.12\sqrt{\frac{EI}{m\,L^4}}$$

2 Problem 2

Assignment 2

A force $p(t) = p_0$ travels across the beam in **Fig. 1** at a uniform velocity v, as shown. Determine an expression for the deflection at midspan as a function of time. Neglect damping and assume the shape function to be

$$\psi(x) = -1 + \cos\left(\frac{2\pi x}{L}\right).$$

Hint: See Example 8.4 in the textbook[1] for guidance on handling a moving load.

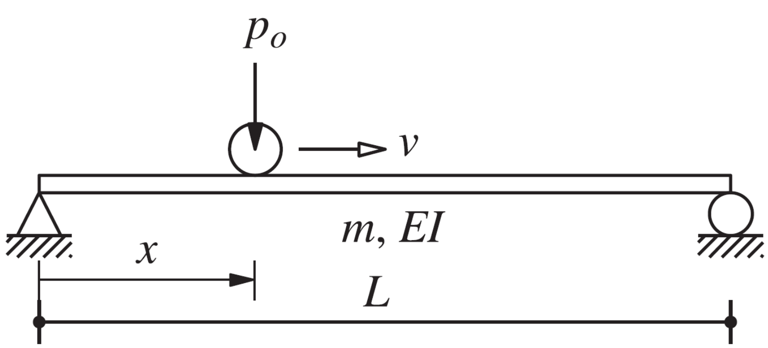


Figure 1: Beam for Problem 2

2.1 Generalized Force Function

The given modal shape corresponds to full cosine cycle, respecting the the boundary conditions that $\psi(0) = \psi(L) = 0$. The definition of the force can be expressed as:

$$p(x,t) = \begin{cases} p_0\delta(x - vt) & 0 \leq t \leq t_d \\ 0 & t \geq t_d \end{cases} \tag{2.1}$$

Where $t_d = L/v$ is the time the load takes to cross the beam, and δ is the Dirac delta function, centered at $x = vt$ (the position of the load at time t). The generalized force is obtained through:

$$\tilde{p} = \int_0^L p(x,t)\psi(x)dx = \begin{cases} -1 + \cos\left(\frac{2\pi vt}{L}\right) & 0 \leq t \leq t_d \\ 0 & t \geq t_d \end{cases} = \begin{cases} -1 + \cos\left(\frac{2\pi t}{t_d}\right) & 0 \leq t \leq t_d \\ 0 & t \geq t_d \end{cases}$$

Where the first clause was derived from the identity $\int_a^b f(x)\delta(x - x_0) = f(x_0)\,(a \leq x \leq b)$[2].

2.2 Equation of Motion

The equation of motion for this undamped generalized sysmte can be described as[1]:

$$\tilde{m}\ddot{z} + \tilde{k}z = \tilde{p}(t) \tag{2.2}$$

Where z is defined as the *generalized displacement* with $u(x,t) = \psi(x)z(t)$ giving the full response of the system at any time at any coordinate x.

2.3 Getting equivalent SDOF Pulse Response

The solution for the EOM presented in **Eq. (2.2)** is derived using Duhamel's integral and can be found in Section C.

2.4 Midspan Evaluation

Setting the modal field as $u(x,t) = \psi(x)\, z(t)$, the midspan value is obtained by evaluating the shape at $x = L/2$. Using $\psi(x) = -1 + \cos(2\pi x/L)$, this becomes

$$\psi\big(\tfrac{L}{2}\big) = -1 + \cos\big(\tfrac{2\pi}{L} \cdot \tfrac{L}{2}\big) = -1 + \cos(\pi) = -2,$$

so the physical displacement at midspan is tied to the generalized coordinate by

$$u\big(\tfrac{L}{2}, t\big) = \psi\big(\tfrac{L}{2}\big)\ z(t) = -2\, z(t).$$

Introducing $\omega_n := \sqrt{\tilde{k}/\tilde{m}}$ for the generalized SDOF and $\Omega := 2\pi/t_d$ for the pulse frequency, the midspan response follows by multiplying the previously derived expressions for $z(t)$ by -2. This preserves the causal structure (driven part while $0 \le t \le t_d$ and free continuation for $t \ge t_d$) and simply scales the amplitude by the mode value at midspan.

2.4.1 During the pulse ($0 \le t \le t_d$)

Substituting $\alpha = t$ into the unified SDOF formula and multiplying by -2 gives the midspan driver/spring split used later:

$$u\big(\tfrac{L}{2}, t\big) = -\frac{2}{\tilde{m}} \left[\frac{\cos(\omega_n t) - 1}{\omega_n^2} + \frac{\cos(\omega_n t) - \cos(\Omega t)}{\Omega^2 - \omega_n^2} \right]. \tag{2.3}$$

The term with ω_n^{-2} tracks the response to the constant offset (-1) in the full-cosine pulse, while the term with $(\Omega^2 - \omega_n^2)^{-1}$ quantifies the dynamic mismatch between the excitation frequency Ω and the natural frequency ω_n.

2.4.2 After the pulse ($t \ge t_d$)

Carrying the integration limit as t_d and rewriting in the phase $t - t_d$ produces a free oscillation whose amplitude and phase are fixed by the pulse history. Specializing to a *full* cosine cycle ($\Omega t_d = 2\pi \Rightarrow \sin(\Omega t_d) = 0$, $\cos(\Omega t_d) = 1$) simplifies the midspan continuation referenced later to

$$u\big(\tfrac{L}{2}, t\big) = -\frac{2}{\tilde{m}} \left[\frac{\cos(\omega_n t) - \cos\big(\omega_n (t - t_d)\big)}{\omega_n^2} + \frac{\cos(\omega_n t) - \cos\big(\omega_n (t - t_d)\big)}{\Omega^2 - \omega_n^2} \right]. \tag{2.4}$$

Both fractions multiply the same cosine difference: the first originates from the offset component of the pulse; the second captures the residual effect of the single-cycle cosine.

2.4.3 Comment on resonance

Setting the pulse frequency equal to the natural frequency turns the mismatch denominators into limits. Taking L'Hôpital on the generalized SDOF formulas and then multiplying by $\psi(L/2) = -2$ gives explicit midspan expressions.

During the pulse $(0 \leq t \leq t_d)$. The midspan response acquires a linearly growing term:

$$u\big(\tfrac{L}{2}, t\big) = -\frac{2}{\tilde{m}}\left[\frac{\cos(\omega_n t) - 1}{\omega_n^2} + \frac{t\,\sin(\omega_n t)}{2\,\omega_n}\right]. \tag{2.5}$$

After the pulse $(t \geq t_d)$. The growth stops at $t = t_d$ and the motion continues as free vibration with amplitude fixed by the accumulated energy:

$$u\big(\tfrac{L}{2}, t\big) = -\frac{2}{\tilde{m}}\left[\frac{\cos(\omega_n t) - \cos\big(\omega_n(t - t_d)\big)}{\omega_n^2} + \frac{t_d\,\sin(\omega_n t)}{2\,\omega_n} - \frac{\sin\big(\omega_n(t - t_d)\big)}{2\,\omega_n}\right]. \tag{2.6}$$

These resonant forms (2.5)–(2.6) correspond to a full cosine cycle when $\Omega t_d = 2\pi$. Under exact resonance, this implies $t_d = 2\pi/\omega_n$, so the linear-in-t build-up lasts exactly one natural period.

Equation (2.5) shows a linearly growing term $t\,\sin(\omega_n t)/(2\omega_n)$: at exact tuning, the pulse injects energy coherently and the midspan amplitude increases over the finite window $0 \leq t \leq t_d$. Once the pulse ends, (2.6) indicates that growth stops and the motion continues as free vibration at ω_n with an amplitude set by the accumulated input, proportional to t_d. The factor -2 reflects the mode value $\psi(L/2)$, fixing the sign and scaling of the midspan response.

3 Problem 3

Assignment 3

A three-story shear frame (rigid beams and flexible columns) in structural steel (E = 29,000 ksi) is shown in the figure below; w = 100 kips, I = 1200 in^4, and the damping ratio is $\zeta = 5\%$. Assuming that the shape function is given by the deflections due to lateral forces equal to the floor weights, determine:

(a) The natural frequency of the structure.

(b) The horizontal floor displacements and story shears due to a ground motion characterized by the design spectrum below, scaled to a peak ground acceleration of $0.6g$.

Among this assignment's problems, the present one adopts a *discrete* generalized SDOF formulation: the shape function is represented by a vector of nodal ordinates rather than a continuous field. A compact side-by-side summary of the correspondence with the continuous case is given in **Fig. 2** [3]:

Term	**Continuous**	**Discrete**
Stiffness	$\tilde{k} = \int_0^L EI(x)\,(\psi(x)'')^2\,dx$	$\tilde{k} = \sum_{j=1}^n k_j(\psi_j - \psi_{j-1})^2$
Mass	$\tilde{m} = \int_0^L m(x)\,(\psi(x))^2\,dx$	$\tilde{m} = \sum_{j=1}^n m_j(\psi_j)^2$
External Force	$\tilde{p} = \int_0^L p(x,t)\psi(x)dx$	$\tilde{p} = \sum_{j=1}^n p_j\psi_j$
Earthquake	$\tilde{L} = \int_0^L m(x)\psi(x)dx$	$\tilde{L} = \sum_{j=1}^n m_j\psi_j$

Figure 2: Generalized SDOF: continuous field vs. discrete vector representation (key mappings summarized).

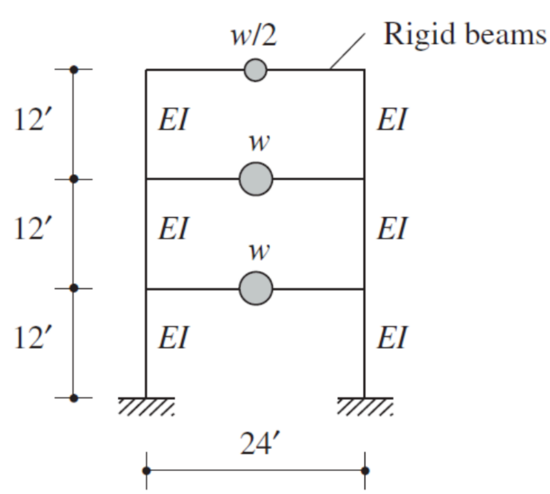


Figure 3: Discrete Floor System

3.1 Generalized system properties

3.1.1 Modal Shape Vector

According to the problem description, the modal shape vector adopted is of the form:

$$\boldsymbol{\psi} = \begin{bmatrix} 1 \\ 1 \\ 0.50 \end{bmatrix}$$

where ψ_i corresponds to the deformation of floor i from top to bottom, and it is taken proportional to the weight distribution as requested in the problem statement.

3.1.2 Generalized Mass

The generalized mass is computed as

$$\tilde{m} = \sum_{i=1}^{3} m_i \psi_i^2 = \frac{1}{g}\left(w + w + \frac{w}{2}\left(\frac{1}{2}\right)^2\right),$$

with $w = 100$ kips for the first two floors and $w/2$ for the third, and $g = 386.4$ in/s^2 (in–kip–s). This gives

$$\tilde{m} = \frac{17}{8}\,\frac{w}{g} = 2.125\,\frac{100}{386.4} = 0.54995 \text{ kip s}^2/\text{in}.$$

3.1.3 Generalized Stiffness

The stiffness of floor i is obtained through classic Euler–Bernoulli beam coefficients for fixed–fixed end conditions [4]:

$$k_i = 24\,\frac{EI}{h^3}.$$

Using the given material and section properties $E = 29{,}000$ ksi, $I = 1200$ in^4, and the story height $h = 12$ ft $= 144$ in,

$$k_i = 24\,\frac{(29{,}000)(1200)}{144^3} = 279.7068 \text{ kip/in} \;=:\; k.$$

Since all stories are identical, $k_i = k$. The generalized stiffness is then obtained as

$$\tilde{k} = \sum_{i=1}^{3} k_i(\psi_i - \psi_{i-1})^2 = k\left([1-0]^2 + [1-1]^2 + [0.50-1]^2\right) = \frac{5}{4}\,k = 349.6335 \text{ kip/in},$$

where $\psi_0 = 0$ has been used at the base.

3.1.4 Determining $\tilde{L}$ and the participation factor Γ

$$\tilde{L} = \sum_{i=1}^{3} m_i \psi_i = \frac{w}{g}\,(1 + 1 + 0.50) = \frac{5}{2}\,\frac{100}{386.4} = 0.64744 \text{ kip s}^2/\text{in}.$$

The modal participation factor is

$$\Gamma = \frac{\tilde{L}}{\tilde{m}} = \frac{20}{17} = 1.17647 \text{ (dimensionless)}.$$

3.2 Natural Frequency

The natural frequency of the generalized system is

$$\omega_n = \sqrt{\frac{\tilde{k}}{\tilde{m}}} = \sqrt{\frac{349.6335}{0.54995}} = 25.214 \text{ s}^{-1}, \qquad T_n = \frac{2\pi}{\omega_n} = 0.2492 \text{ s}.$$

3.3 Equation of Motion

The generalized equation of motion

$$\ddot{z} + 2\zeta\omega_n \dot{z} + \omega_n^2 z = -\Gamma\, \ddot{u}_g \tag{3.1}$$

governs the problem. A damping ratio of $\zeta = 0.05$ is given for the structure, so that the spectrum provided in **??** can be used.

3.4 Spectral response for $\widehat{S}_a = 2.71$ at $0.60g$

Scaling the dimensionless spectral ordinate to $0.60g$ gives the physical spectral acceleration

$$S_a = 0.60\, \widehat{S}_a\, g = 0.60 \times 2.71 \times 386.4 \;=\; 628.29 \text{ in/s}^2.$$

The peak generalized displacement follows from

$$z_{\max} = \frac{\Gamma\, S_a}{\omega_n^2} = \frac{(1.17647)(628.29)}{(25.214)^2} = 1.163 \text{ in}.$$

Floor peak displacements (mode-scaled):

$$u_1^{\max} = \psi_1 z_{\max} = 1.163 \text{ in}, \qquad u_2^{\max} = 1.163 \text{ in}, \qquad u_3^{\max} = 0.5\, z_{\max} = 0.581 \text{ in}.$$

3.5 Story drifts

:

$$\Delta_1 = u_1 - 0 = 1.163 \text{ in}, \qquad \Delta_2 = u_2 - u_1 = 0 \text{ in}, \qquad \Delta_3 = u_3 - u_2 = -0.581 \text{ in}.$$

3.6 Equivalent lateral forces (modal)

$$F_i = \Gamma\, m_i\, \psi_i\, S_a \quad \Rightarrow \quad \begin{aligned} F_1 &= \Gamma \frac{w}{g}(1)\, S_a = 190.242 \text{ kip}, \\ F_2 &= \Gamma \frac{w}{g}(1)\, S_a = 190.242 \text{ kip}, \\ F_3 &= \Gamma \frac{w/2}{g}(0.5)\, S_a = 47.56 \text{ kip}. \end{aligned}$$

Hence the story shears by cumulative sum of floor forces are

$$V_3 = F_3 = 47.56 \text{ kip}, \qquad V_2 = F_2 + F_3 = 237.80 \text{ kip}, \qquad V_1 = F_1 + F_2 + F_3 = 428.04 \text{ kip}.$$

Which corresponds with the total shear force on the structure.

A Appendix — Elastic Curve for the span and the overhang

A.1 Geometry and notation

Let l be the distance between supports, a the overhang length to the right of the right support, and P a downward point load at the free end of the overhang. The beam is prismatic with constant E, I. The coordinates used are

$$x \in [0, l] \text{ (left support to right support)}, \qquad x_1 \in [0, a] \text{ (right support to free end)}.$$

Downward deflection is taken as positive.

A.2 Governing equation

$$\frac{d^2 y}{dx^2} = \frac{M}{EI}.$$

A.3 Bending moments

For a point load at the free end, the internal bending moments are linear in each region:

$$M_s(x) = \frac{Pa}{l}\,x, \qquad 0 \le x \le l \quad \text{(between supports)},$$
$$M_o(x_1) = P\,(a - x_1), \qquad 0 \le x_1 \le a \quad \text{(overhang)}.$$

A.4 Elastic curve between supports $0 \le x \le l$

Integrating twice gives

$$y_s'' = \frac{Pa}{EI\,l}x \;\Rightarrow\; y_s' = \frac{Pa}{2EI\,l}x^2 + C_1, \qquad y_s = \frac{Pa}{6EI\,l}x^3 + C_1 x + C_2.$$

Simply supported ends imply $y_s(0) = 0 \Rightarrow C_2 = 0$ and $y_s(l) = 0$, hence

$$0 = \frac{Pa}{6EI\,l}l^3 + C_1 l \;\Rightarrow\; C_1 = -\frac{Pa\,l}{6EI}.$$

Therefore,

$$\boxed{y_s(x) = \frac{Pa\,x}{6EI\,l}\left(l^2 - x^2\right), \qquad 0 \le x \le l.}$$

The slope at the right support is

$$y_s'(l) = \frac{Pa}{2EI\,l}l^2 - \frac{Pa\,l}{6EI} = \frac{Pa\,l}{3EI}.$$

A.5 Elastic curve on the overhang $0 \le x_1 \le a$

Two integrations yield

$$y_o'' = \frac{P}{EI}(a - x_1) \;\Rightarrow\; y_o' = \frac{P}{EI}\left(ax_1 - \tfrac{1}{2}x_1^2\right) + C_3, \qquad y_o = \frac{P}{EI}\left(\tfrac{a}{2}x_1^2 - \tfrac{1}{6}x_1^3\right) + C_3x_1 + C_4.$$

Continuity of deflection and slope at the right support ($x = l \Leftrightarrow x_1 = 0$) gives

$$y_o(0) = y_s(l) = 0 \Rightarrow C_4 = 0, \qquad y_o'(0) = y_s'(l) = \frac{Pa\,l}{3EI} \Rightarrow C_3 = \frac{Pa\,l}{3EI}.$$

Hence,

$$\boxed{y_o(x_1) = \frac{P\,x_1}{6EI}\Big(2al + 3ax_1 - x_1^2\Big), \qquad 0 \le x_1 \le a.}$$

A.6 Summary (piecewise) with change of variables

Introducing the global coordinate on the entire right half of the beam as

$$X \in [0, l+a], \qquad X = \begin{cases} x, & 0 \le X \le l, \\ l + x_1, & l \le X \le l + a, \end{cases} \qquad \Rightarrow \qquad x_1 = X - l \text{ on the overhang,}$$

and substituting gives

$$y(X) = \begin{cases} \dfrac{Pa\,X}{6EI\,l}\left(l^2 - X^2\right), & 0 \le X \le l, \\ \dfrac{P\,(X-l)}{6EI}\Big(2al + 3a\,(X-l) - (X-l)^2\Big), & l \le X \le l + a. \end{cases}$$

This expression coincides with the closed-form result shown in the reference image.

A.7 Specialization

Setting $l = 3L$ and $a = L$ gives

$$y_s(x) = \frac{P\,x}{18EI}\left(9L^2 - x^2\right), \qquad 0 \le x \le 3L,$$

$$y_o(x_1) = \frac{P\,x_1}{6EI}\left(6L^2 + 3L\,x_1 - x_1^2\right), \qquad 0 \le x_1 \le L.$$

Single global coordinate. Introducing $X \in [0, 4L]$ with

$$X = \begin{cases} x, & 0 \le X \le 3L, \\ 3L + x_1, & 3L \le X \le 4L, \end{cases} \qquad \Rightarrow \qquad x_1 = X - 3L \text{ on the overhang,}$$

and substituting produces

$$y(X) = \begin{cases} \dfrac{P\,X}{18EI}\left(9L^2 - X^2\right), & 0 \le X \le 3L, \\ \dfrac{P\,(X-3L)}{6EI}\Big(6L^2 + 3L\,(X-3L) - (X-3L)^2\Big), & 3L \le X \le 4L. \end{cases}$$

Simplified piecewise form ($l = 3L,\ a = L$)

Using the change of variables $X \in [0, 4L]$ with $X = x$ on $[0, 3L]$ and $X = 3L + x_1$ on $[3L, 4L]$, the deflection simplifies to

$$y(X) = \begin{cases} \dfrac{P}{18EI}\, X\big(9L^2 - X^2\big) = \dfrac{P}{18EI}\, X\,(3L - X)(3L + X), & 0 \le X \le 3L, \\ \dfrac{P}{6EI}\,(3L - X)\,\big(X^2 - 9LX + 12L^2\big), & 3L \le X \le 4L. \end{cases}$$

B Appendix — Deriving $\tilde{m}$ and $\tilde{k}$ from $\Psi(X)$

This appendix derives the generalized mass $\tilde{m}$ and stiffness $\tilde{k}$ associated with the piecewise shape function $\Psi(X)$. The evaluation proceeds by splitting the integrals at $X = 3L$, expanding the resulting polynomials, integrating term by term, and assembling the contributions. Final expressions are presented in a compact form to fit the page margins.

B.1 shape function and definitions.

$$\Psi(X) = \begin{cases} \dfrac{P}{18EI}\, X\left(9L^2 - X^2\right), & 0 \le X \le 3L, \\ \dfrac{P}{6EI}\,(3L - X)\left(X^2 - 9LX + 12L^2\right), & 3L \le X \le 4L, \end{cases}$$

$$\tilde{m} \;=\; m\int_0^{4L} \Psi^2\, dX, \qquad \tilde{k} \;=\; EI\int_0^{4L} \left(\Psi''\right)^2 dX.$$

B.2 Generalized mass $\tilde{m}$

$$\tilde{m} = m\left[\int_0^{3L}\left(\frac{P}{18EI}X(9L^2-X^2)\right)^2 dX \;+\; \int_{3L}^{4L}\left(\frac{P}{6EI}(3L-X)(X^2-9LX+12L^2)\right)^2 dX\right].$$

For $0 \le X \le 3L$:

$$\left(\frac{P}{18EI}X(9L^2-X^2)\right)^2 = \frac{P^2}{324E^2I^2}\left(81L^4X^2 - 18L^2X^4 + X^6\right),$$

$$\int_0^{3L} \cdots\, dX = \frac{P^2}{324E^2I^2}\left[27L^4X^3 - \frac{18}{5}L^2X^5 + \frac{X^7}{7}\right]_0^{3L} = \frac{18}{35}\,\frac{P^2L^7}{E^2I^2}.$$

For $3L \le X \le 4L$:

$$(3L-X)(X^2-9LX+12L^2) = -X^3 + 12LX^2 - 39L^2X + 36L^3,$$

$$\int_{3L}^{4L}\left(\frac{P}{6EI}\cdots\right)^2 dX = \frac{19}{35}\,\frac{P^2L^7}{E^2I^2}.$$

Combining:

$$\boxed{\tilde{m} = m\left(\frac{18}{35} + \frac{19}{35}\right)\frac{P^2L^7}{E^2I^2} = \frac{37}{35}\,\frac{m\,P^2L^7}{E^2I^2}}$$

B.3 Generalized stiffness $\tilde{k}$

For $0 \le X \le 3L$:

$$\Psi_1(X) = \frac{P}{18EI}\,(9L^2X - X^3), \qquad \Psi_1''(X) = -\,\frac{P}{3EI}\,X,$$

$$\int_0^{3L}\left[\Psi_1''(X)\right]^2 dX = \frac{P^2}{9E^2I^2}\int_0^{3L} X^2 dX = \frac{P^2L^3}{E^2I^2}.$$

For $3L \le X \le 4L$:

$$\Psi_2(X) = \frac{P}{6EI}\,(-X^3 + 12LX^2 - 39L^2X + 36L^3), \qquad \Psi_2''(X) = \frac{P}{EI}\,(4L - X),$$

$$\int_{3L}^{4L}\big[\Psi_2''(X)\big]^2 dX = \frac{P^2}{E^2I^2}\int_{3L}^{4L}(4L-X)^2 dX = \frac{1}{3}\,\frac{P^2L^3}{E^2I^2}.$$

Multiplying by EI and adding:

$$\boxed{\tilde{k} = EI\left(\frac{P^2L^3}{E^2I^2} + \frac{1}{3}\frac{P^2L^3}{E^2I^2}\right) = \frac{4}{3}\,\frac{P^2L^3}{EI}}$$

B.4 Ratio $\tilde{k}/\tilde{m}$

$$\boxed{\frac{\tilde{k}}{\tilde{m}} = \frac{\frac{4}{3}\,\frac{P^2L^3}{EI}}{\frac{37}{35}\,\frac{mP^2L^7}{E^2I^2}} = \frac{140}{111}\,\frac{EI}{m\,L^4}}$$

C Appendix — Response to Full Cosine Pulse

C.1 Preface

This appendix derives the displacement $u(t)$ of an undamped SDOF oscillator with mass m and stiffness k under a full-cycle cosine pulse of duration t_d. The natural circular frequency is $\omega_n = \sqrt{k/m}$. The generalized load (already projected by ψ) is

$$\tilde{p}(t) = \begin{cases} -1 + \cos(\Omega t), & 0 \le t \le t_d, \\ 0, & t \ge t_d, \end{cases} \qquad \Omega := \frac{2\pi}{t_d},$$

this becomes a causal input on $[0, t_d]$ consisting of a static offset (-1) plus a single-cycle cosine at frequency Ω. The undamped impulse response is

$$h(t-\tau) = \frac{1}{m\,\omega_n} \sin\big(\omega_n(t-\tau)\big).$$

Inserting these in Duhamel's integral and capping the upper limit at $\alpha := \min(t, t_d)$ to encode causality directly, the response becomes

$$u(t) = \frac{1}{m\,\omega_n} \int_0^{\alpha} \big[-1 + \cos(\Omega\tau)\big]\, \sin\big(\omega_n(t-\tau)\big)\, d\tau.$$

For readability, the integral is decomposed into a "constant" contribution from the offset -1 and an "oscillatory" contribution from $\cos(\Omega\tau)$; each part is integrated in closed form and then assembled.

C.2 Integral decomposition and evaluation

Define

$$I_1 := -\int_0^{\alpha} \sin\big(\omega_n(t-\tau)\big)\, d\tau, \qquad I_2 := \int_0^{\alpha} \cos(\Omega\tau)\, \sin\big(\omega_n(t-\tau)\big)\, d\tau,$$

so that $u(t) = \dfrac{1}{m\,\omega_n}\,(I_1 + I_2)$.

Constant part I_1

Introducing $s = t - \tau$ $(ds = -d\tau)$ maps $\tau : 0 \to \alpha$ into $s : t \to t - \alpha$. The integral becomes

$$I_1 = \int_{t-\alpha}^{t} \sin(\omega_n s)\, ds = \frac{\cos\big(\omega_n(t-\alpha)\big) - \cos(\omega_n t)}{\omega_n},$$

which captures the effect of the static offset -1.

Oscillatory part I_2

Expanding the shifted sine,

$$\sin\big(\omega_n(t-\tau)\big) = \sin(\omega_n t)\cos(\omega_n\tau) - \cos(\omega_n t)\sin(\omega_n\tau),$$

and using the product–to–sum identities, the required primitives over $[0, \alpha]$ are

$$\int_0^\alpha \cos(\Omega\tau)\cos(\omega_n\tau)\,d\tau = \tfrac{1}{2}\left[\frac{\sin((\Omega-\omega_n)\alpha)}{\Omega-\omega_n} + \frac{\sin((\Omega+\omega_n)\alpha)}{\Omega+\omega_n}\right],$$

$$\int_0^\alpha \cos(\Omega\tau)\sin(\omega_n\tau)\,d\tau = \tfrac{1}{2}\left[\frac{1-\cos((\omega_n+\Omega)\alpha)}{\omega_n+\Omega} + \frac{1-\cos((\omega_n-\Omega)\alpha)}{\omega_n-\Omega}\right].$$

Substituting yields

$$\begin{aligned} I_2 = {} & \frac{\sin(\omega_n t)}{2}\left[\frac{\sin((\Omega-\omega_n)\alpha)}{\Omega-\omega_n} + \frac{\sin((\Omega+\omega_n)\alpha)}{\Omega+\omega_n}\right] \\ & - \frac{\cos(\omega_n t)}{2}\left[\frac{1-\cos((\omega_n+\Omega)\alpha)}{\omega_n+\Omega} + \frac{1-\cos((\omega_n-\Omega)\alpha)}{\omega_n-\Omega}\right]. \end{aligned}$$

C.3 Unified expression

Combining I_1 and I_2, the response becomes

$$\begin{aligned} u(t) = {} & \frac{1}{m\,\omega_n^2}\Big[\cos(\omega_n t) - \cos\big(\omega_n(t-\alpha)\big)\Big] \\ & + \frac{1}{2m\,\omega_n}\Bigg\{\sin(\omega_n t)\left[\frac{\sin((\Omega-\omega_n)\alpha)}{\Omega-\omega_n} + \frac{\sin((\Omega+\omega_n)\alpha)}{\Omega+\omega_n}\right] \\ & \qquad - \cos(\omega_n t)\left[\frac{1-\cos((\omega_n+\Omega)\alpha)}{\omega_n+\Omega} + \frac{1-\cos((\omega_n-\Omega)\alpha)}{\omega_n-\Omega}\right]\Bigg\}, \qquad \alpha = \min(t, t_d). \end{aligned} \tag{C.1}$$

C.4 Specializing the time window

During the pulse $0 \le t \le t_d$ ($\alpha = t$)

Algebra on (C.1) produces

$$u(t) = \frac{1}{m}\left[\frac{\cos(\omega_n t) - 1}{\omega_n^2} + \frac{\cos(\omega_n t) - \cos(\Omega t)}{\Omega^2 - \omega_n^2}\right]. \tag{C.2}$$

The first fraction is the response to the constant offset; the second is the dynamic mismatch term between Ω and ω_n.

After the pulse $t \ge t_d$ ($\alpha = t_d$)

Carrying the upper limit as t_d and rewriting in the phase $t - t_d$ gives

$$\begin{aligned} u(t) = \frac{1}{m}\Bigg[& \frac{\cos(\omega_n t) - \cos\big(\omega_n(t-t_d)\big)}{\omega_n^2} \\ & + \frac{\Omega\sin(\Omega t_d)\,\sin\big(\omega_n(t-t_d)\big) - \omega_n\cos(\Omega t_d)\,\cos\big(\omega_n(t-t_d)\big) + \omega_n\cos(\omega_n t)}{\Omega^2-\omega_n^2}\Bigg]. \end{aligned} \tag{C.3}$$

For a *full* cosine cycle, $\Omega t_d = 2\pi$ makes $\sin(\Omega t_d) = 0$ and $\cos(\Omega t_d) = 1$, reducing (C.3) to a compact combination of $\cos(\omega_n t)$ and $\cos\big(\omega_n(t-t_d)\big)$.

C.5 Comment on Reasonance

Setting the pulse frequency equal to the natural frequency, $\Omega = \omega_n$, turns the mismatch denominators into singular limits. Taking L'Hôpital gives finite expressions with a *linearly growing* term while the pulse is on.

Resonant form during the pulse $0 \le t \le t_d$

$$u(t) = \frac{1}{m}\left[\frac{\cos(\omega_n t) - 1}{\omega_n^2} + \frac{t\,\sin(\omega_n t)}{2\,\omega_n}\right]. \tag{C.4}$$

The term $t\,\sin(\omega_n t)/(2\omega_n)$ expresses the energy accumulation due to perfect tuning ($\Omega = \omega_n$).

Resonant form after the pulse $t \ge t_d$

$$u(t) = \frac{1}{m}\left[\frac{\cos(\omega_n t) - \cos\big(\omega_n (t - t_d)\big)}{\omega_n^2} + \frac{t_d\,\sin(\omega_n t)}{2\,\omega_n} - \frac{\sin\big(\omega_n (t - t_d)\big)}{2\,\omega_n}\right]. \tag{C.5}$$

The response continues as a free oscillation at ω_n with an amplitude and phase fixed by the resonant build-up during $0 \le t \le t_d$. The linear-in-time term switches to a constant coefficient $t_d/(2\omega_n)$ once the pulse ends, as expected from energy input over a finite resonant window.

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