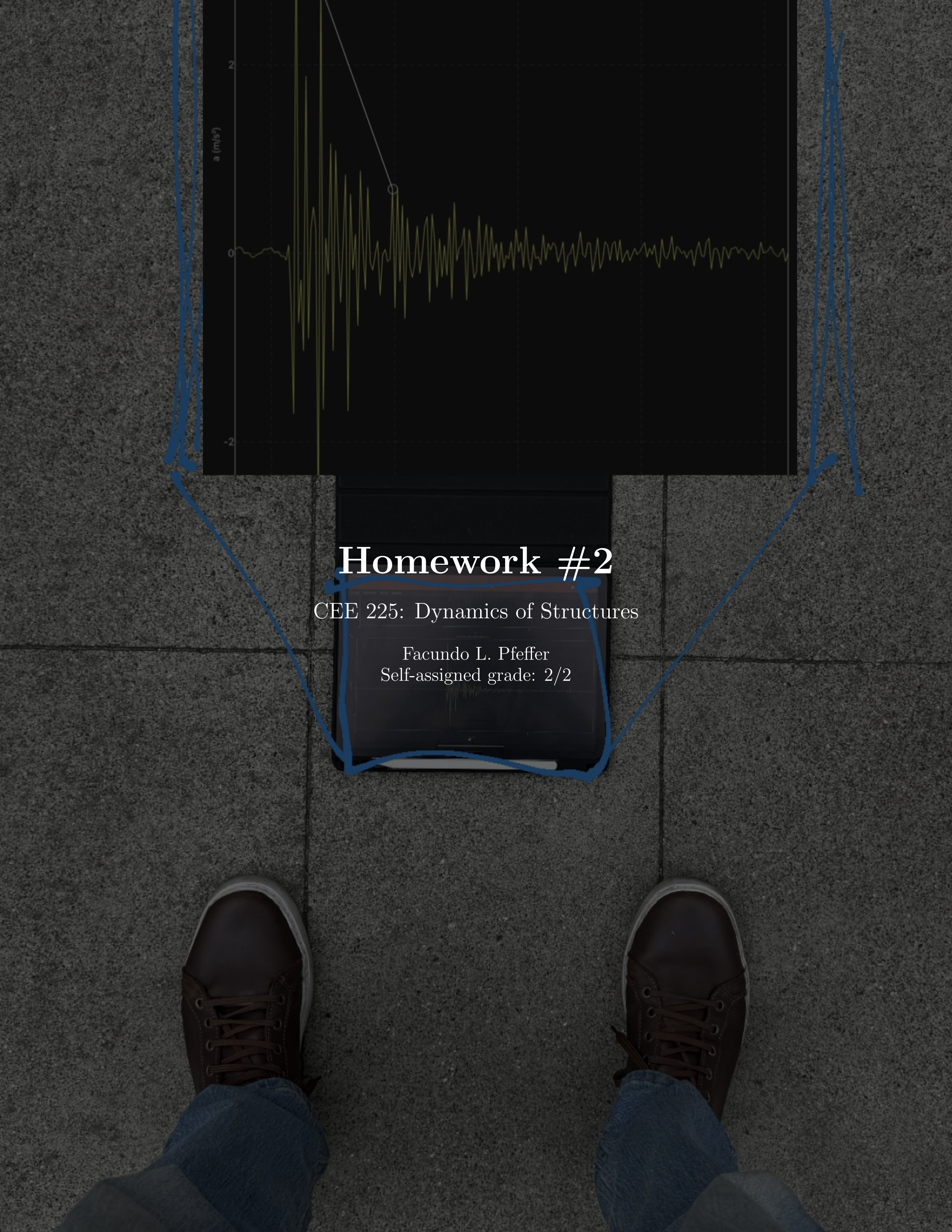

Homework #2

CEE 225: Dynamics of Structures

Facundo L. Pfeffer
Self-assigned grade: 2/2

Based on the guidelines provided, **I self-assign a grade of 2/2 for this homework**. I have made a legitimate effort to complete 100% of the problems.

1 Assignment 1

Assignment 1

A mass m is initially at rest, partially supported by a spring and partially by stops (see Figure 1). In the position shown, the spring force is mg/4. At time t = 0 the stops are rotated, suddenly releasing the mass. Determine the motion of the mass.
Main concept: definition of the coordinate system and relation with the form of the equation of motion.

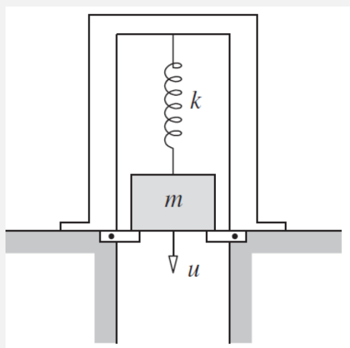

Figure 1: Single DOF system for Assignment 1.

1.1 Introduction

The coordinate system is defined such that the downward direction is positive. The static equilibrium position u_{st} corresponds to the point where the spring force balances the weight of the mass (when the stop is released). At this position:

$$ku_{st} = mg \quad \Rightarrow \quad u_{st} = \frac{mg}{k}$$

Let $u(t)$ denote the dynamic displacement measured from the static equilibrium position. Therefore, the total displacement from the unstretched spring configuration is:

$$u_{total}(t) = u_{st} + u(t)$$

After the stops are released at $t = 0$, the mass is subjected only to the restoring force of the spring. In terms of the variable $u(t)$, the equation* of motion is:

$$m\ddot{u}(t) + ku(t) = 0$$

This corresponds to the classical free vibration differential equation of an undamped single Degree-of-freedom (DOF) system [1][2]. The general solution is:

$$
\begin{aligned}
u(t) &= A\cos(\omega_n t) + B\sin(\omega_n t), \\
\dot{u}(t) &= -A\omega_n \sin(\omega_n t) + B\omega_n \cos(\omega_n t), \\
\ddot{u}(t) &= -A\omega_n^2 \cos(\omega_n t) - B\omega_n^2 \sin(\omega_n t) = -\omega_n^2 u(t)
\end{aligned} \tag{1}
$$

Where A and B are constants to be determined by the initial conditions and $\omega_n = \sqrt{\frac{k}{m}}$ the natural frequency of the single DOF system.

1.2 Initial Conditions

The initial conditions are determined based on the initial spring force and the definition of $u(t)$ as the displacement from the static equilibrium position. At $t = 0$, the spring was already stretched to support a force of $\frac{mg}{4}$. When the stops are released, the weight of the mass becomes fully effective, but the spring has not yet adjusted. The resulting initial displacement from the new equilibrium is:

$$u(0) = -\frac{3}{4}u_{\text{st}} = -\frac{3}{4} \cdot \frac{mg}{k} = -\frac{3mg}{4k}$$

Furthermore, the mass is initially at rest, so:

$$\dot{u}(0) = 0$$

Applying these initial conditions to the general solution **Eq. (1)**:

$$
\begin{aligned}
&u(0) = A\cos(0) + B\sin(0) = A \\
&\Rightarrow A = -\frac{3mg}{4k} \\
&\dot{u}(t) = -A\omega_n \sin(\omega_n t) + B\omega_n \cos(\omega_n t) \\
&\dot{u}(0) = -A\omega_n \cdot 0 + B\omega_n \cdot 1 = B\omega_n \\
&\Rightarrow B = 0
\end{aligned}
$$

1.3 Equation of Motion (EOM)

Therefore, the Equation of Motion that satisfies the initial conditions is:

$$u(t) = -\frac{3mg}{4k}\cos\left(\sqrt{\frac{k}{m}}t\right)$$

This describes the oscillatory motion of the mass about the static equilibrium position, with amplitude $\frac{3mg}{4k}$ and natural frequency $\omega_n = \sqrt{\frac{k}{m}}$.

1.4 Displacement Plot

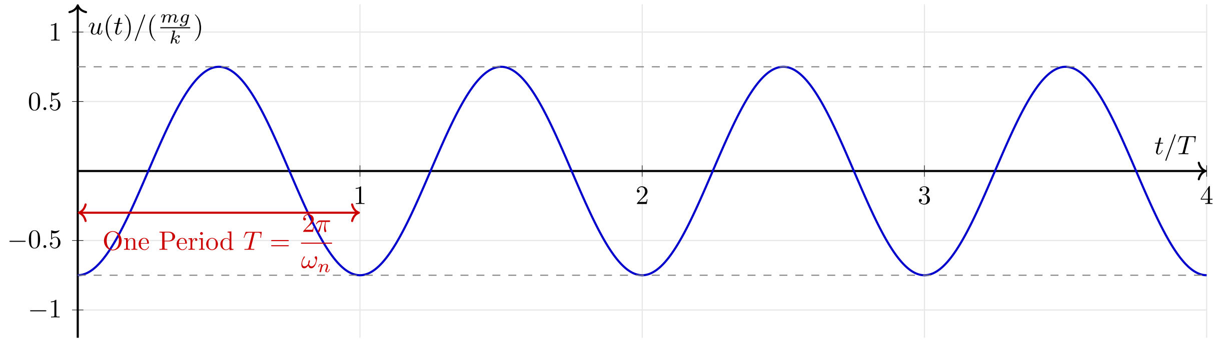

Figure 2: Time response of the system measured from static equilibrium u_{st}.

2 Problem 2

Assignment 2

The weight of the wooden block shown in Fig. 2 is 20 lb and the spring stiffness is 120 lb/in. The block is initially at rest. A bullet weighing 0.4 lb is fired at a speed of 50 ft/sec into the block and becomes embedded in the block. Determine the resulting motion $u(t)$ of the block.

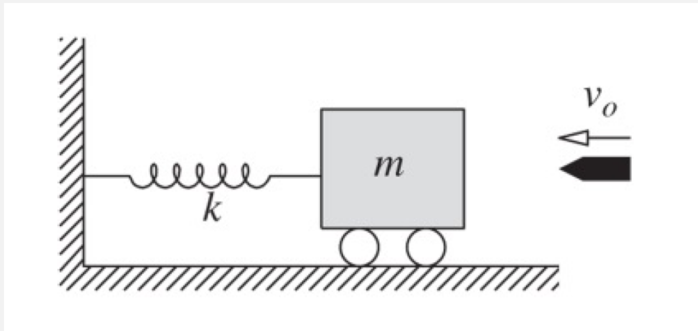

Figure 3: Single DOF system for Assignment 2

2.1 Introduction

The system consists of a spring–mass oscillator initially at rest. A bullet of mass m_b is fired horizontally at speed v_b and becomes embedded in a wooden block of mass m, forming a single composite mass $m + m_b$ immediately after the impact.

This interaction is modeled as a perfectly **plastic collision**, in which the two bodies stick together after contact. According to the principle of conservation of linear momentum (a direct consequence of Newton's Second Law for systems with no external horizontal force), the total momentum before and after impact must be the same. As a result, the velocity of the combined mass immediately after the collision is:

$$m_b v_b = (m + m_b)\dot{u}(0) \quad \Rightarrow \quad \dot{u}(0) = \frac{m_b}{m + m_b} v_b$$

Because the duration of the collision is extremely short, the force exchanged during impact is large but acts over a negligible time interval. As such, the impulse from the bullet instantaneously imparts a finite velocity to the block without allowing the spring to compress during the interaction. Therefore, the collision is treated as an *instantaneous event*, and the resulting velocity $\dot{u}(0)$ is considered the initial condition for the subsequent motion of the system.

Once the impact has occurred, the bullet-block system behaves as a single mass $M = m + m_b$ oscillating under the action of the linear spring. Since no external forces act on the system in the horizontal direction after the collision, Newton's Second Law governs the free vibration of the mass:

$$M\ddot{u}(t) + ku(t) = 0$$

Here, $u(t)$ denotes the horizontal displacement of the mass from the spring's unstretched position, and k is the spring stiffness. The initial velocity $\dot{u}(0)$ results from the collision, while the initial displacement is zero.

Assigning the positive direction of $u(t)$ to the right, the resulting motion of the system is governed by the same homogeneous differential equation as in **Eq. (1)**. Therefore, the general solution retains the same functional form, but with constants A and B determined by the new initial conditions specific to this problem:

$$\begin{aligned} u(t) &= A\cos(\omega_n t) + B\sin(\omega_n t), \\ \dot{u}(t) &= -A\omega_n \sin(\omega_n t) + B\omega_n \cos(\omega_n t), \\ \ddot{u}(t) &= -\omega_n^2 u(t) \end{aligned}$$

where $\omega_n = \sqrt{\dfrac{k}{M}}$ is the natural frequency of the combined mass system.

2.2 Initial Conditions

At $t = 0$, the impact occurs and causes the block-bullet system to move with velocity $\dot{u}(0)$, but with no initial spring displacement:

$$\begin{aligned} u(0) &= 0 \\ \dot{u}(0) &= -\frac{m_b}{m + m_b} v_b \end{aligned}$$

Substituting into the general solution:

$$\begin{aligned} &u(0) = A\cos(0) + B\sin(0) = A \quad \Rightarrow \quad A = 0 \\ &\dot{u}(0) = B\omega_n \quad \Rightarrow \quad B = \frac{m_b v_b}{(m + m_b)\omega_n} \end{aligned}$$

2.3 Numerical Substitution

Given:

$$m_b = 0.4\,\text{lb}, \quad v_b = 50\,\text{ft/s} = 600\,\text{in/s}, \quad m = 20\,\text{lb}, \quad k = 120\,\text{lb/in}$$

The initial velocity is:

$$\dot{u}(0) = \frac{0.4\,\text{lb}}{20.4\,\text{lb}} \cdot 600\,\frac{\text{in}}{\text{s}} = 11.76\,\frac{\text{in}}{\text{s}}$$

Assuming mass is kept in lb (approximated), the natural frequency is:

$$\omega_n = \sqrt{\frac{\overbrace{120\,\text{lb/in}}^{\approx 21021\,\text{N/m}}}{\underbrace{20.4\,\text{lb}}_{\approx 9.25\,\text{kg}}}} \approx 47.67\,\text{rad/s}$$

Then:

$$B = -\frac{11.76\,\frac{\text{in}}{\text{s}}}{47.67\,\frac{\text{rad}}{\text{s}}} = -0.247\,\text{in}$$

2.4 Equation of Motion

The derived EOM takes the form:

$$u(t) = 0.247\,\sin(47.67\,t) \quad [\text{in}]$$

This describes the oscillatory motion of the mass after impact. The amplitude is determined by the imparted momentum and system mass. A plot is decipted in **Fig. 4**.

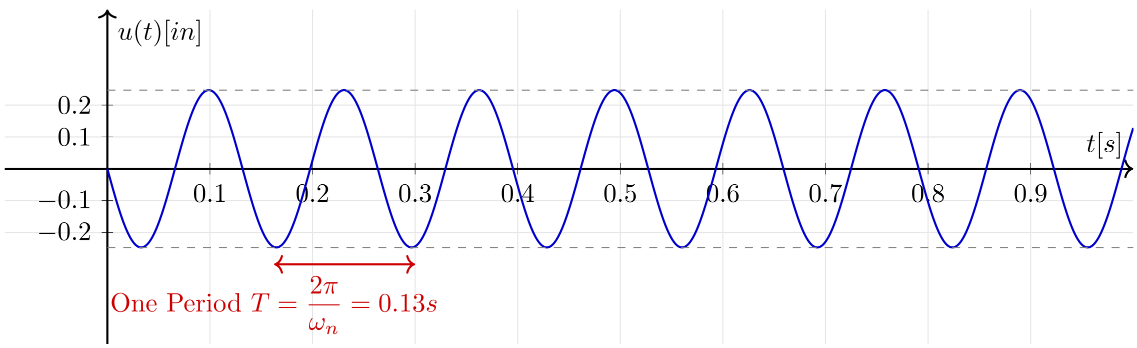

Figure 4: Time response of the system measured from static equilibrium u_{st}.

3 Problem 3

Assignment 3

The vertical suspension system of an automobile is idealized as a viscously damped SDF system. Under the 3500-lb weight of the car, the suspension system deflects 2.1 inches. The suspension is designed to have a damping ratio of $\zeta = 0.7$ with no one in the car.

(a) Calculate the damping and stiffness coefficients of the suspension.

(b) With four 150-lb passengers in the car, what is the effective damping ratio?

(c) Calculate the natural vibration frequency for case (b).

Main concept: static equilibrium position and the vibration properties of a system with damping.

3.1 Damping and Stiffness Determination on Mass Change

The suspension stiffness can be directly computed from the static deflection under the vehicle's own weight. Given that the car weighs 3500 lb and causes a vertical deflection of 2.1 in, the equivalent linear spring constant is:

$$k = \frac{3500\,\text{lb}}{2.1\,\text{in}} = 1666.67\,\text{lb/in} = 29.26\,\text{kN/m}$$

The corresponding natural frequency of the system, assuming undamped motion and converting weight to mass using $m = \frac{W}{g}$ with $g = 386.09\,\text{in/s}^2$, is:

$$\omega_n' = \sqrt{\frac{k}{m}} = 13.56\,\text{rad/s}$$

Given that the damping ratio of the suspension system is specified as $\zeta' = 0.70$, the damping coefficient c can be obtained from its definition:

$$\zeta' = \frac{c}{2m\omega_n'} \quad \Rightarrow \quad c = 66440\,\frac{\text{lb}}{\text{s}}$$

Now, considering the addition of four passengers (each 150 lb), the total weight becomes $W = 4100\,\text{lb}$. This increased mass leads to a new natural frequency:

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{1666.67\,\text{lb/in} \cdot 386.09\,\text{in/s}^2}{4100\,\text{lb}}} = 12.52\,\text{rad/s}$$

Finally, the updated damping ratio is computed using the previously calculated damping coefficient:

$$\zeta = \frac{c}{2M\omega_n} = \frac{66440}{2 \cdot 4100 \cdot 12.52} = 0.647$$

4 Problem 4

Assignment 4

Find a linear elastic oscillator, measure its damped natural frequency and determine its percentage of critical damping. Submit a sketch of the system, your measurements, and calculations.

For this assignment, the selected linear elastic oscillator is the pedestrian bridge connecting Davis Hall and Sutardja Dai Hall to Cory Hall at the University of California, Berkeley. As shown in **Fig. 5**, the degree of freedom (DOF) under consideration corresponds to vertical motion, and the structure is idealized as a single-degree-of-freedom (SDOF) system. The idealized element follows a curved trajectory that represents the geometric locus of successive center-of-mass positions of the bridge, which is reasonably assumed to lie within the main deck, where the majority of the structural mass is concentrated.

The experimental setup is illustrated in **Fig. 6** and **Fig. 7**. Excitation is induced by jumping vertically at the midspan of the bridge, generating a transient impulse. The resulting vertical acceleration is measured using the Phyphox mobile application[3], installed on an iPad Air.

It is important to note that, similar to the excitation mechanism described in **Section 2**, the impulsive nature of the jump imposes an initial vertical velocity on the system, which governs the subsequent free vibration response.

With the results of the vertical acceleration observed in **Fig. 8**, using the local maxima between 15 periods we can obtain the damped natural frequency ω_d:

$$T = \frac{\overbrace{15T}^{measured}}{15} = \frac{0.39s}{15} = 0.026w \Rightarrow \omega_d = 2\pi/T = 241\frac{rad}{s}$$

4.1 Estimation of Damping Ratio by Logarithmic Decrement

The steady-state free vibration response of the system can be reasonably modeled as a single degree-of-freedom oscillator:

$$u(t) = u_0 e^{-\zeta\omega_n t}\sin(\omega_d t + \phi), \tag{2}$$

where u_0 is the displacement amplitude, ζ the damping ratio, ω_n the natural frequency, $\omega_d = \omega_n\sqrt{1-\zeta^2}$ the damped natural frequency, and ϕ the phase angle. Since the maxima of the sine function correspond to $\sin(\cdot) = \pm 1$, the envelope of the peaks is given by:

$$u_{\text{peak}}(t) = u_0 e^{-\zeta\omega_n t}. \tag{3}$$

Taking the ratio between two peaks separated by j cycles with period $T_D = \frac{2\pi}{\omega_d}$:

$$\frac{u_i}{u_{i+j}} = \frac{e^{-\zeta\omega_n t_i}}{e^{-\zeta\omega_n(t_i+jT_D)}} = e^{\zeta\omega_n jT_D}.$$

Taking natural logarithms:

$$\ln\left(\frac{u_i}{u_{i+j}}\right) = \frac{2\pi\zeta j}{\sqrt{1-\zeta^2}} \approx 2\pi\zeta j \quad \text{(for } \zeta < 0.2\text{)}.$$

Thus, the damping ratio is estimated by the **logarithmic decrement method**:

$$\zeta = \frac{1}{2\pi j}\ln\left(\frac{u_i}{u_{i+j}}\right), \quad \zeta < 0.2. \tag{4}$$

Application to the Measured Data

In this case, the first peak ($i = 0$) and the peak after $j = 15$ cycles are considered. Since the measurements are obtained as accelerations, the amplitudes must be converted to displacement amplitudes using:

$$u = \frac{\ddot{u}}{\omega_d^2}.$$

From the recorded accelerations:

$$\ddot{u}_0 = 3.56\,\frac{m}{s^2} \quad \Rightarrow \quad u_0 = 0.0613\,\text{mm},$$

$$\ddot{u}_{15} = 0.68\,\frac{m}{s^2} \quad \Rightarrow \quad u_{15} = 0.0117\,\text{mm}.$$

Finally, substituting into the logarithmic decrement formula:

$$\zeta = \frac{1}{2\pi(15)}\ln\left(\frac{u_0}{u_{15}}\right) = 0.018 = 1.8\%.$$

According to different studies conducted on pedestrian bridges [4], the value obtained by this simple experiment lies in what is considered a an expected behavior.

Figure 5: Idealized linear elastic oscillator selected for study. Photo taken at UC Berkeley between Davis and Cory Hall on 2025-09-11.

Figure 6: Overall experiment setup. The iPad Air equipped with the Phyphox accelerometer application is placed on the bridge deck at midspan.

Figure 7: Closer view of the instrumentation. The iPad Air running *Phyphox* [3] is shown laying horizontally to the deck surface for vibration recording.

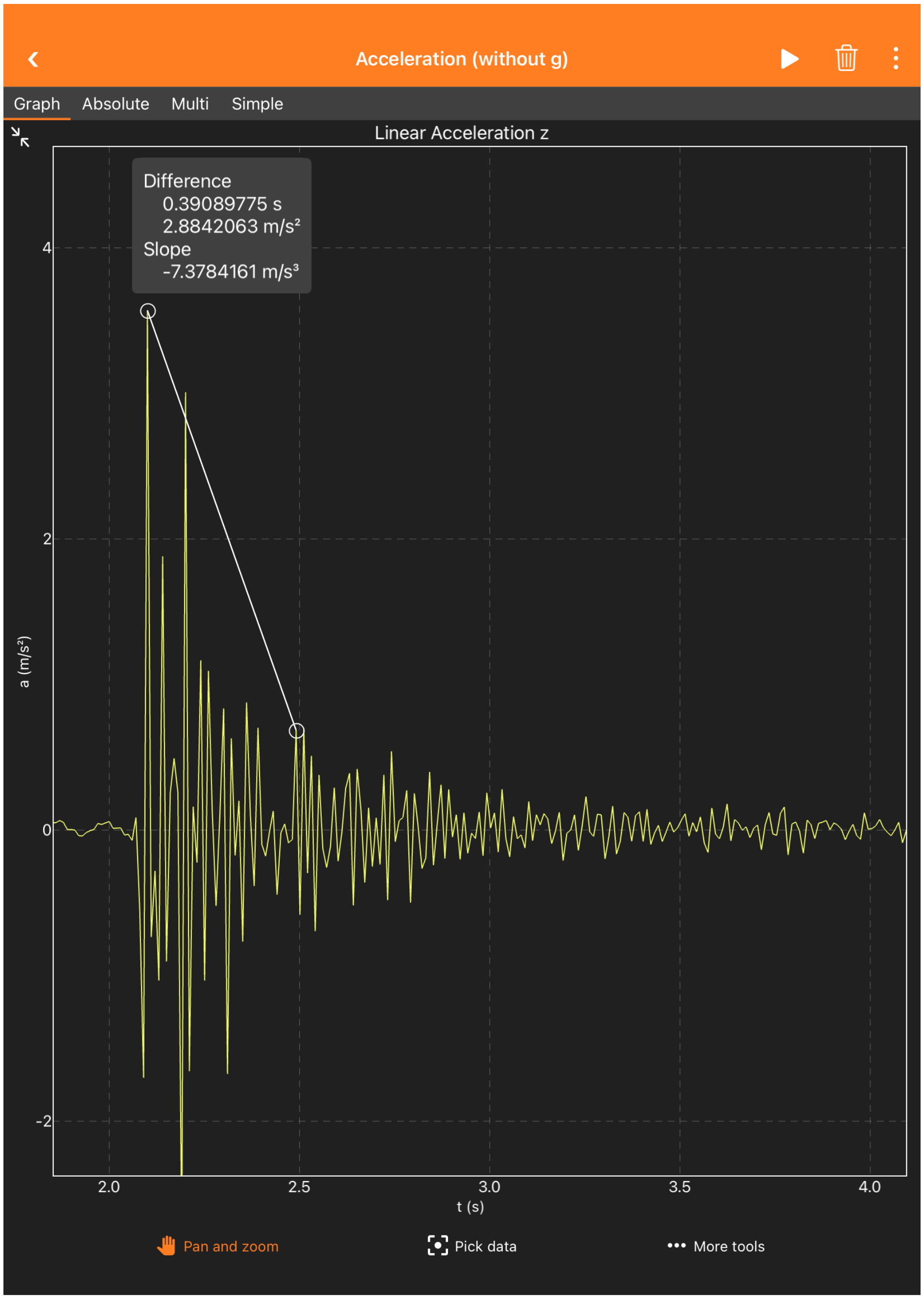

Figure 8: Time history of vertical acceleration measured by *Phyphox* [3] during free vibration of the bridge.

5 References

[1] A. K. Chopra, *Dynamics of Structures: Theory and Applications to Earthquake Engineering.* Boston: Pearson Education, 4th, global edition ed., 2014.

[2] M. DeJong, F. Filippou, S. Govindjee, A. D. Kiureghian, K. Mosalam, J. Moehle, and E. Opabola, "Semm graduate program primer: 2025," SEMM Reports Series UCB/SEMM-2025/04, University of California, Berkeley, July 2025.

[3] S. Staacks, S. Hütz, H. Heinke, and C. Stampfer, "Advanced tools for smartphone-based experiments: phyphox," *Physics Education*, vol. 53, no. 4, p. 045009, 2018.

[4] S. Castellanos-Toro, M. Marmolejo, J. Marulanda, A. Cruz, and P. Thomson, "Frequencies and damping ratios of bridges through operational modal analysis using smartphones," *Construction and Building Materials*, vol. 188, pp. 490–504, 2018.