Self-Grading Declaration

Based on the guidelines provided, **I self-assign a grade of 2/2 for this homework**. I have made a legitimate effort to complete 100% of the problems.

0 Preface: Fundamental Hypothesis

The fundamental assumptions from which the governing equations will be derived are the following:

1. **Plane sections remain plane under deformation.** This is known as the Euler-Bernoulli hypothesis and implies that shear distortion is neglected.
2. **Material linearity is preserved** throughout all stages of loading. That is, stress is linearly proportional to strain within the elastic regime.
3. **Displacements are relatively small**, which allows the use of linearized kinematics[1] and static equilibrium equations written in the undeformed configuration.

These hypothesis set up a framework for the calculations where the relation between displacements and forces is **linear** [1, 2].

[1] Specifficaly for this assignment, this hypothesis will repeteadly take place in the derivations.

1 Problem 1

Assignment 1

Consider the truss model shown below.

(i) Number the free DOFs and element basic deformations using our convention. Then set up the kinematics equations that relate the element deformations to the nodal displacements,

$$\boldsymbol{V} = \boldsymbol{A_f U_f}. \tag{1}$$

(ii) If elements b and c are heated up by $\Delta T = 150$ degrees, determine and draw the deformed shape of the structure. The axial strain due to heating is uniform and given by $\alpha\Delta T$, where α is the coefficient of thermal expansion, 5×10^{-6} in this problem.

1.1 Problem Setup

The free degrees-of-freedom of the sysmtem are depicted in **Fig. 1**:

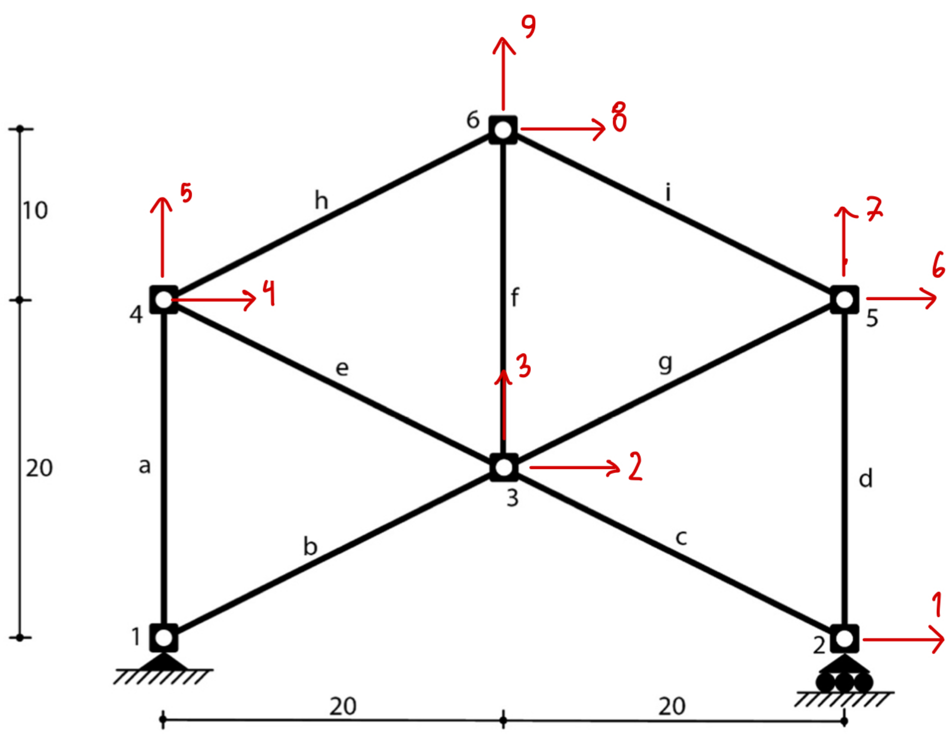

Figure 1: Free DOFs for Problem 1

The kinematics matrix of **Eq. (1)** can be derived from analyzing how different truss elements will stretch under unitary displacement of each of the free DOFs. The resulting matrix for the depicted structure is:

$$\boldsymbol{A_f} = \frac{1}{\sqrt{5}} \begin{bmatrix} 0 & 0 & 0 & 0 & \sqrt{5} & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{5} & 0 & 0 \\ 0 & 2 & -1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{5} & 0 & 0 & 0 & 0 & 0 & \sqrt{5} \\ 0 & -2 & -1 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & -1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & -1 & -2 & 1 \end{bmatrix}$$

1.2 Heating effect

For isotropic materials, the thermal strain caused by a uniform temperature increase is expressed as:

$$\varepsilon_T = \alpha \, \Delta T, \tag{2}$$

where α is the coefficient of linear thermal expansion and ΔT is the temperature change [**? ?**]. This relation follows from linear thermoelasticity, assuming constant α, small deformations, and uniform temperature.

Given:

$$\alpha = 5 \times 10^{-6} \,/^{\circ}\mathrm{C}, \qquad \Delta T = 150^{\circ}\mathrm{C},$$

the thermal strain is

$$\varepsilon_T = \alpha \, \Delta T = (5 \times 10^{-6})(150) = 7.5 \times 10^{-4}. \tag{3}$$

Each heated element therefore elongates freely by

$$\Delta L = \varepsilon_T L_a = (7.5)(10\sqrt{5}) = 1.67 \, 10^{-2} \tag{4}$$

producing a small overall rotation and expansion in the truss structure. The corresponding deformation vector for this is:

$$\boldsymbol{V} = 1.67 \times 10^{-4} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Thus the corresponding displacements can be obtained using **Eq. (1)** and taking the inverse of $\boldsymbol{A_f}$. The calculations are carried out in Python (see Section 4) and the resulting displacements vector $\boldsymbol{U}$ is:

$$\boldsymbol{U} = \boldsymbol{A_f}^{-1}\boldsymbol{V} = \begin{bmatrix} 0.0375 \\ 0.01875 \\ 0.0000 \\ 0.01875 \\ 0.0000 \\ 0.01875 \\ 0.0000 \\ 0.01875 \\ 0.0000 \end{bmatrix}$$

1.3 Deformed Body

The deformed body is depicted in **Fig. 2**

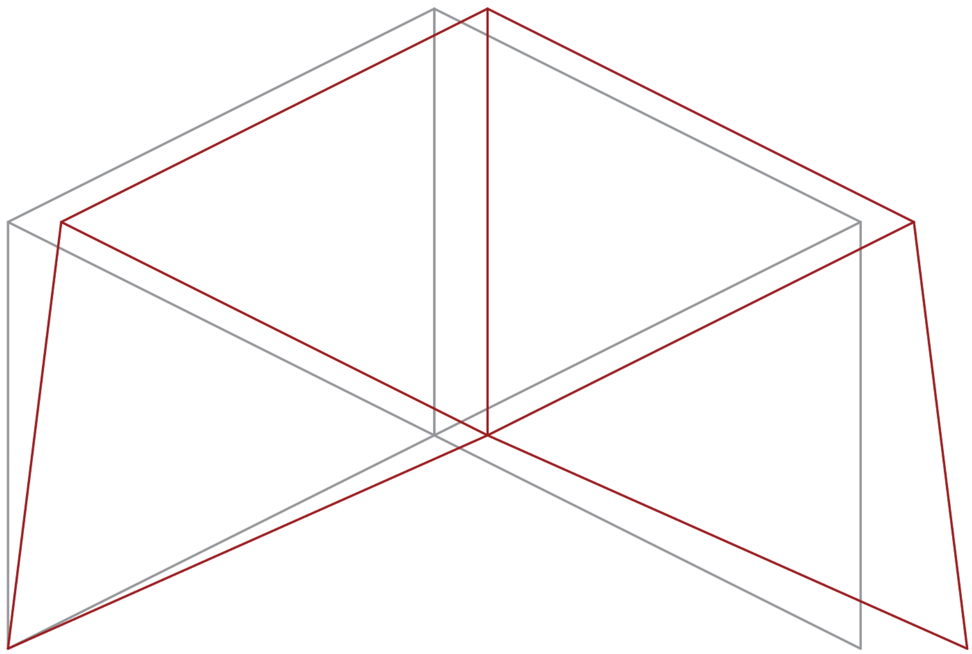

Figure 2: Deformed Body. Scaled.

2 Problem 2

Assignment 2

Element b of the beam shown below is subjected to a temperature gradient that makes it sad.

(i) Using the minimum number of free DOFs (i.e., exclude all axial and trivial rotational DOFs), determine the kinematic matrix $\boldsymbol{A_f}$. You may check your result for the $\boldsymbol{A_f}$ matrix by comparing it against the transpose of the $\boldsymbol{B_f}$ matrix obtained in Homework Assignment 2, Problem 3(ii).

(ii) If the curvature in element b is $\kappa = -3 \times 10^{-3}\,(\text{unit length})^{-1}$, determine the nodal displacements $\boldsymbol{U_f}$.

(iii) Determine the rotations at nodes 1 and 5 and the hinge rotation next to node 4.

(iv) Draw the deformed shape.

2.1 Kinematic Matrix Setup

The non-trivial and non-axial free displacement DOFs associated with the structure are depicted in **Fig. 4**:

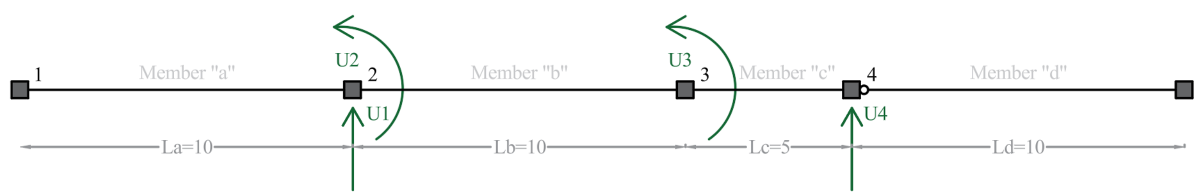

Figure 3: Non-trivial and non-axial free displacement DOFs for structure in Problem 2.

The corresponding kinematics matrix is:

$$\boldsymbol{A_f} = B_f^{\mathsf{T}} = \begin{bmatrix} -\frac{1}{10} & 1 & 0 & 0 \\ \frac{1}{10} & 1 & 0 & 0 \\ \frac{1}{10} & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{5} \end{bmatrix}$$

2.2 Introduction of Curvature

The curvature of a member, according to the hypothesis detailed in Section 0, can be approximated as:

$$\kappa = \frac{d\vartheta}{dx} \tag{5}$$

Where κ is the curvature of the member. In this problem, the constant value of $\kappa = -3\,10^{-3}$ is given. Note that the negative sign indicates that the beam will deflect with its tension fiber in the top of the beam. The resulting deformations can be calculated using the set of equations in **Eq. (6)** for the starting node i and end node j for constant curvature:

$$V_{\varepsilon_i} = -\frac{1}{L}\int_0^L (L-x)\,dx = \frac{1}{2}\kappa L, \tag{6a}$$

$$V_{\varepsilon_j} = \frac{1}{L}\int_0^L (L-x)\,dx = \frac{1}{2}\kappa L. \tag{6b}$$

Thus:

$$V_{\varepsilon_2} = +0.015$$
$$V_{\varepsilon_3} = -0.015$$

The resulting deformation vector is then:

$$V = \begin{bmatrix} 0 \\ 0.015 \\ -0.015 \\ 0 \end{bmatrix}$$

Further, the displacement field is obtained through **Eq. (1)**:

$$U = \begin{bmatrix} 0.075 \\ 0.0075 \\ -0.0225 \\ -0.1125 \end{bmatrix}$$

2.3 Node rotations

The missing rotations are recovered:

$$\vartheta(x=0) = \frac{U_1}{10} = 0.0075$$
$$\vartheta(x=35) = -\frac{U_4}{10} = 0.01125$$

To obtain the hinge rotation in node 4, the rotation on the right end of member c and left end of element d are first computed:

$$V_j^c = \frac{U_4}{L_c} = \frac{-0.1125}{5} = -0.0225 \qquad V_i^d = -\frac{U_4}{L_d} = -\frac{-0.1125}{10} = 0.01125$$

Finally

$$V_{h4} = V_j^c - V_i^d = -0.0225 - 0.01125 = -0.03375$$

The final drawing is depicted in **Fig. 4**:

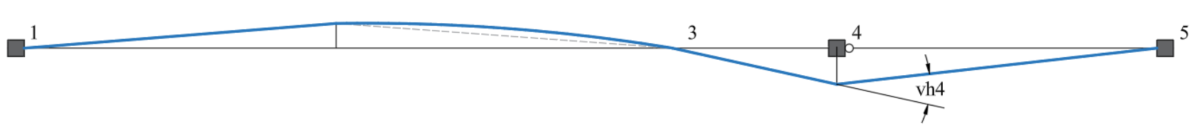

Figure 4: Deformed shape for Problem 2.

3 Problem 3

Assignment 3

Consider the structural model shown below, where a, b, and c are inextensible frame elements subjected to a uniform curvature of $\kappa = +0.001\,(\text{unit length})^{-1}$, while d is a truss element that has an installation length that is 0.02 units longer than specified in the drawing.

(i) Set up $\boldsymbol{V} = \boldsymbol{A_f}\boldsymbol{U_f}$ for the smallest possible number of DOFs.

(ii) Determine $\boldsymbol{U_f}$ and the hinge rotations at nodes 1, 3, and the node rotation at 4.

(iii) Draw the deformed shape.

3.1 Kinematics Matrix Setup

The free degrees of freedom considered are depicted in **Fig. 5**

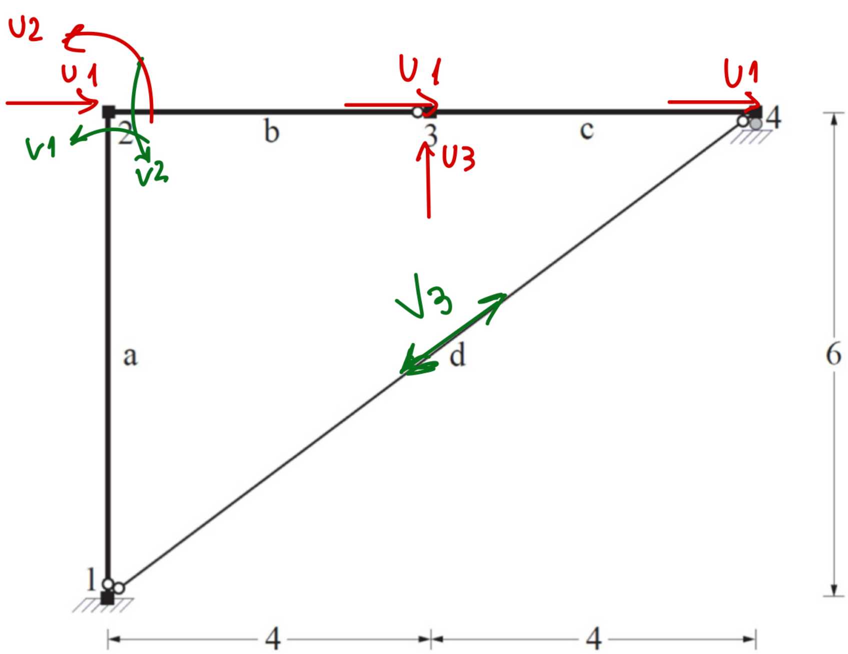

Figure 5: Free degrees of freedom for Problem 3.

The corresponding kinematics matrix is

$$\boldsymbol{A_f} = \begin{bmatrix} \frac{1}{6} & 1 & 0 \\ 0 & 1 & -\frac{1}{4} \\ \frac{4}{5} & 0 & 0 \end{bmatrix}$$

3.2 Displacement Field

The deformations vector, using the curvature definition in **Eq. (1)** can be computed as:

$$v = \frac{1}{2}\kappa$$

$$\boldsymbol{V} = \begin{bmatrix} \frac{1}{2}\kappa L_a \\ -\frac{1}{2}\kappa L_b \\ 0.02 \end{bmatrix} = \begin{bmatrix} 0.003 \\ -0.002 \\ 0.02 \end{bmatrix}$$

The displacement field

$$\boldsymbol{U} = \boldsymbol{A_f}^{-1}\boldsymbol{V} = \begin{bmatrix} 0.025 \\ -1.166\,10^{-3} \\ 3.333\,10^{-3} \end{bmatrix}$$

The hinge rotations at nodes 1,3, and node rotation at node 4 is:

$$v_{h1} = -\frac{U_1}{6} + \left(-\frac{1}{2}\kappa L_a\right) = -\frac{0.025}{6} - \frac{1}{2}(0.001)(6) = -7.16\,10^{-3}$$

Analogously:

$$v_{h3} = (v_{\varepsilon 2} + \frac{U_3}{L_b}) - (-v_{\varepsilon_c} - \frac{U_3}{L_c}) = (0.002 + \frac{3.33\,10^{-3}}{4}) - (-0.002 + \frac{3.33\,10^{-3}}{4}) = 0.00566$$

Where $v_{\varepsilon_c} = 0.002$ is the curvature deformation for element c

$$v_4 = -\frac{U_3}{L_c} + v_{\varepsilon_c} = -\frac{3.33\,10^{-3}}{4} + 0.002 = 0.00116$$

3.3 Deformed Shape

Finally, the (scaled) deformed shaped of the body is depicted in **Fig. 6**.

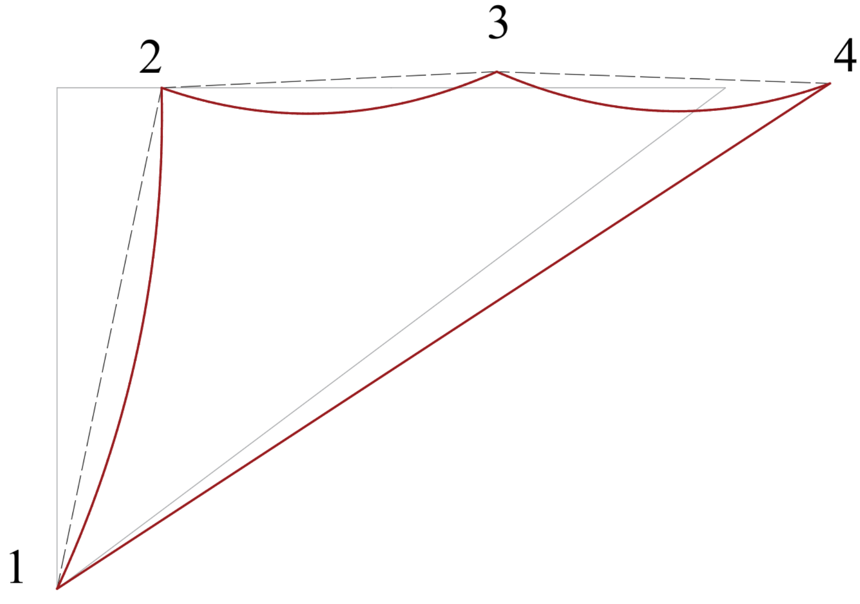

Figure 6: Deformed shape of the body. Scaled.

4 Supporting Python Script

A supporting Python script was used to verify manual results and avoid performing cumbersome calculations by hand. The latest version can be found at UCBerkeley-SEMM-MS-Codebook[2], within the *CEE 220* directory. For this script, the only needed dependency is numpy[3].

Python 3.13 Code

```
import numpy as np

def run_assignment_1():

    Af = (1 / np.sqrt(5)) * np.array([
        [0, 0, 0, 0, np.sqrt(5), 0, 0, 0, 0],
        [0, 2, 1, 0, 0, 0, 0, 0, 0],
        [2, -2, 1, 0, 0, 0, 0, 0, 0],
        [0, 0, 0, 0, 0, 0, np.sqrt(5), 0, 0],
        [0, 2, -1, -2, 1, 0, 0, 0, 0],
        [0, 0, -np.sqrt(5), 0, 0, 0, 0, 0, np.sqrt(5)],
        [0, -2, -1, 0, 0, 2, 1, 0, 0],
        [0, 0, 0, -2, -1, 0, 0, 2, 1],
        [0, 0, 0, 0, 0, 2, -1, -2, 1]
    ])

    # Thermal expansion
    alpha = 5e-6  # coefficient of thermal expansion [1/ C ]
    delta_T = 150  # temperature change [ C ]
    L = 10 * np.sqrt(5)  # element lengthn
    delta_L = alpha * delta_T * L

    V = np.array([
        0,
        delta_L,
        delta_L,
        0,
        0,
        0,
        0,
        0,
        0
    ]).reshape(-1, 1)
    print(V)
    U = np.linalg.inv(Af) @ V
    print("U =\n", U)


if __name__ == "__main__":
    run_assignment_1()
```

[2] https://github.com/Facundo-Pfeffer/UCBerkeley-SEMM-MS-Codebook/tree/master/CEE220_StructuralAnalysis

5 References

[1] F. C. Filippou, *Structural Analysis: Theory and Applications*. University of California, Berkeley, n.d. Department of Civil and Environmental Engineering.

[2] M. DeJong, F. Filippou, S. Govindjee, A. D. Kiureghian, K. Mosalam, J. Moehle, and E. Opabola, “Semm graduate program primer: 2025,” SEMM Reports Series UCB/SEMM-2025/04, University of California, Berkeley, July 2025.

[3] C. R. Harris, K. J. Millman, S. J. van der Walt, R. Gommers, P. Virtanen, D. Cournapeau, E. Wieser, J. Taylor, S. Berg, N. J. Smith, R. Kern, M. Picus, S. Hoyer, M. H. van Kerkwijk, M. Brett, A. Haldane, J. F. del Río, M. Wiebe, P. Peterson, P. Gérard-Marchant, K. Sheppard, T. Reddy, W. Weckesser, H. Abbasi, C. Gohlke, and T. E. Oliphant, “Array programming with NumPy,” *Nature*, vol. 585, pp. 357–362, Sept. 2020.